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Schwinger, Pegg and Barnett approaches and a relationship between angular and Cartesian quantum descriptions

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Abstract
From development of an original idea due to Schwinger, it is shown that it is possible to recover, from the quantum description of a degree of freedom characterized by a finite number of states (i.e., without a classical counterpart), the usual canonical variables of position/momentum *and* angle/angular momentum, the latter appearing, perhaps surprisingly, as a limit of the first.

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1. Introduction

Quantum mechanics has a lot of intriguing aspects. Particularly prominent among these aspects is the fact that there are still a handful of fundamental questions which remain matters of debate after so many years. Among these questions is the problem of the quantum phase, which a few years ago saw an important chapter (but not the final one, it seems) in its history, triggered by the advent of the approach due to Pegg and Barnett (PB) [1].

Within the broad grasp of the PB formalism, there is the particular and important problem of one-dimensional angular coordinates in quantum mechanics. This specific problem is less problematic than the question of the phase as a whole, but nevertheless it is 'solved' (or re-solved) within the procedure of PB. In this paper, I shall relate the PB approach, in this particular context, to an idea presented by Schwinger; from this relation, although relatively simple, there seem to emerge quite interesting results.

Schwinger's original idea was to recover a usual Cartesian degree of freedom (e.g., a degree of freedom endowed with a canonically related pair of observables of position and linear momentum) from a degree of freedom described by a finite set of states (that is, without a classical counterpart) through a limiting process [2]. Here, I extend his discussion, showing that the Cartesian degree of freedom can in fact be recovered by an infinite number of limiting processes. The relationship referred to above comes from noting that a limiting element of those infinitely many processes which work for the Cartesian case reproduces exactly the PB approach for the angle/angular momentum case. So, in this sense, a circle would be the limit of a line and not the opposite.

There is a conceptual bonus in the Schwinger procedure for obtaining the quantum description of a Cartesian degree of freedom. Schwinger's approach to finite and discrete degrees of freedom is, at its root, by nature built upon quantum mechanical concepts: quantum state, incompatible observables and unitary transformations. Once, starting from this, one has obtained the quantum description of degrees of freedom with classical counterparts, it is as explicit as possible that there must necessarily be no quantization of classical quantities involved in such descriptions. The PB approach to angular coordinates can also be seen from the same perspective and therefore shares this virtue. If one sees the two descriptions (Cartesian and angular) as different manifestations of the same situation, then there might be room for new interpretations of the ultimate physical meaning of such mathematical structures.

2. The Schwinger unitary operator bases and the discrete genesis of the canonical variables

A long time ago, Schwinger noticed that one can obtain a complete basis in operator space from a pair of unitary operators U and V, which act on each others sets of N eigenvectors as follows:

$$V^s|u_n\rangle = |u_{n-s}\rangle \qquad U^s|v_n\rangle = |v_{n+s}\rangle \qquad n = 0, 1, \ldots, N-1 \tag{1}$$

where cyclic notation is understood, i.e.

$$|u_k\rangle \equiv |u_{k(\mathrm{mod}\,N)}\rangle \qquad |v_m\rangle \equiv |v_{(m\,\mathrm{mod}\,N)}\rangle. \tag{2}$$

The operators have the roots of unity as eigenvalues:

$$U|u_k\rangle = \exp\left[\frac{2\pi\mathrm{i}}{N}k\right]|u_k\rangle \qquad V|v_k\rangle = \exp\left[\frac{2\pi\mathrm{i}}{N}k\right]|v_k\rangle \tag{3}$$

and therefore

$$U^N = V^N = \hat{1}. \tag{4}$$

The pair also obey Weyl algebra:

$$U^jV^l = \exp\left[\frac{2\pi\mathrm{i}}{N}jl\right]V^lU^j \tag{5}$$

and its eigenvectors are connected by a discrete Fourier transform:

$$\langle v_k|u_n\rangle = \frac{1}{\sqrt{N}}\exp\left[-\frac{2\pi\mathrm{i}}{N}kn\right] \tag{6}$$

which means that the two sets of states carry a maximum degree of incompatibility. It must be made clear that this construction is absolutely general, as Schwinger *obtains* all results above from the mere existence of a complete family (with a finite number) of eigenstates of a given abstract operator.

Schwinger has realized that the pair of operators $\{U, V\}$ could be used to define a basis in *operator* space (as will be discussed in more detail in following work [3]) and has also noticed that, if ones goes from this discrete finite-dimensional case to a usual continuous degree of freedom, the ordinary position–momentum description is recovered.

To further extend Schwinger's original idea (which he concisely explored in just a few lines), first we must introduce a scaling factor:

$$\epsilon = \sqrt{\frac{2\pi}{N}} \tag{7}$$

which will become infinitesimal as $N \to \infty$. Then, two Hermitian operators $\{P, Q\}$ (for simplicity, odd Ns will be considered, as even values require only a little more care and a heavier notation):

$$P = \sum_{j=-(N-1)/2}^{(N-1)/2} j\epsilon^{\delta} p_0 |v_j\rangle\langle v_j| \qquad Q = \sum_{j'=-(N-1)/2}^{(N-1)/2} j'\epsilon^{2-\delta} q_0 |u_{j'}\rangle\langle u_{j'}| \tag{8}$$

are constructed out of the projectors of the eigenstates of U and V. δ is a free parameter which might assume any value in the open interval (0, 2) (the range of the original Schwinger discussion is equivalent to setting $\delta = 1$). $\{p_0, q_0\}$ are real parameters that might carry units of momentum and position, respectively, and $\epsilon^{\delta} p_0$ and $\epsilon^{2-\delta} q_0$ are the distances between successive eigenvalues of the P- and Q-operators. With the help of these, we can rewrite the Schwinger operators as

$$V = \exp\left[\frac{\mathrm{i}\epsilon^{2-\delta} P}{p_0}\right] \qquad U = \exp\left[\frac{\mathrm{i}\epsilon^{\delta} Q}{q_0}\right]. \tag{9}$$

Also let both eigenstate sets be relabelled as

$$|v_j\rangle \equiv |p\rangle \qquad |u_{j'}\rangle = |q\rangle \qquad \text{with } q = q_0\epsilon^{2-\delta} j' \text{ and } p = p_0\epsilon^{\delta} j. \tag{10}$$

With this,

$$P = \sum_{p=-[(N-1)/2]\epsilon^{\delta} p_0}^{[(N-1)/2]\epsilon^{\delta} p_0} p|p\rangle\langle p| \qquad Q = \sum_{q=-[(N-1)/2]\epsilon^{2-\delta} q_0}^{[(N-1)/2]\epsilon^{2-\delta} q_0} q|q\rangle\langle q| \tag{11}$$

and equations (1) now read

$$\exp\left[\frac{\mathrm{i}p' Q}{p_0 q_0}\right] |p\rangle = |p + p'\rangle \tag{12}$$

and

$$\exp\left[\frac{\mathrm{i}q' P}{p_0 q_0}\right] |q\rangle = |q - q'\rangle \tag{13}$$

if $\{p', q'\}$ are defined following the recipe of (10).

The equations above have a clear analogy with the usual relations between position and momentum, apart from the fact that only discrete values of the parameters are allowed and that the cyclic conditions (equation (2)) still hold.

The $N \to \infty$ limit can now be easily taken. For δ assuming any value in the open interval (0, 2), each Hermitian operator defined in equations (11) will feature an unbounded and continuous spectrum, as the limit leads them to[1]

$$P = \int_{-\infty}^{\infty} p|p\rangle\langle p|\,\mathrm{d}p \qquad Q = \int_{-\infty}^{\infty} q|q\rangle\langle q|\,\mathrm{d}q \tag{14}$$

and equations (12), (13) will now be valid for any real numbers $\{p, q, p', q'\}$. It must be observed that, because of the way in which they are obtained, the labels $\{p, q\}$ span the set of all rational numbers, which is a proper subset of the set of real numbers. On the other hand, every real number can be written as the limit of an infinite sequence of rational numbers. Then the expression

$$\exp\left[\frac{\mathrm{i}(p' + p'' + p''' + \cdots)Q}{p_0 q_0}\right] |p\rangle = |p + p' + p'' + p''' + \cdots\rangle \tag{15}$$

[1] This limit has to be taken carefully, but it works as if the limiting value of the discrete projector $|p\rangle\langle p|$ is, after taking the limit, $|p\rangle\langle p|\,\mathrm{d}p$. Just consider what happens to the resolution of unity to see this.

might converge to any real eigenvalue and its associated eigenvector. This is enough to ensure that the whole usual Hilbert space of usual canonical variables is recovered[2] [4]. Also, after the limit is taken the cyclic condition becomes irrelevant, and the familiar relations are easily recovered from their discrete counterparts:

$$Q|q\rangle = q|q\rangle \qquad \langle q'|q\rangle = \delta\left(q'-q\right) \qquad -\infty \leqslant q', q \leqslant \infty \tag{16}$$

$$P|p\rangle = p|p\rangle \qquad \langle p'|p\rangle = \delta(p'-p) \qquad \langle p|q\rangle = \frac{1}{\sqrt{2\pi p_0 q_0}} \exp\left(\frac{\mathrm{i}pq}{p_0 q_0}\right). \tag{17}$$

Therefore the results for a degree of freedom endowed with a usual position–momentum canonical pair of variables are completely reproduced, provided that the product of the parameters $p_0 q_0$ is set to $\hbar$.

The $\epsilon^{2-\delta}$- and ϵ^{δ}-factors, roughly speaking, control how 'fast' (as N increases) one will become unable to identify the distance between labels of consecutive eigenvalues. The result above is then rather peculiar, as it states that *how* you take this limit does not affect the final result. The usual canonical variables would be recovered anyway.

But things can become different if you consider the extreme situation $\delta = 0$ (or $\delta = 2$, which is equivalent). In this case one of the variables is not scaled at all and what follows is almost identical to the PB scheme (for simplicity, the reference angle is set to zero). One would have

$$V = \exp\left[\frac{\mathrm{i}\epsilon^2 M}{m_0}\right] \qquad U = \exp\left[\frac{\mathrm{i}\Theta}{\theta_0}\right] \tag{18}$$

where

$$M = \sum_{j=-(N-1)/2}^{(N-1)/2} j m_0 |v_j\rangle\langle v_j| \qquad \Theta = \sum_{j'=-(N-1)/2}^{(N-1)/2} \epsilon^2 j' \theta_0 |u_{j'}\rangle\langle u_{j'}|. \tag{19}$$

If desired, the exponential of the angle operator might be used instead of the operator itself, for the well known reasons given in [5]. The pair $\{m_0, \theta_0\}$ may carry different dimensional units. Let (again) both eigenstates sets be relabelled as

$$|v_j\rangle \equiv |m\rangle \qquad |u_{j'}\rangle = |\theta\rangle \qquad \text{with } \theta = \theta_0 \epsilon^2 j' \text{ and } m = m_0 j. \tag{20}$$

In the $N \to \infty$ limit one would have

$$M = \sum_{m=-\infty}^{\infty} m|m\rangle\langle m| \qquad \Theta = \int_{-\pi}^{\pi} \theta|\theta\rangle\langle\theta|\,\mathrm{d}\theta \tag{21}$$

$$\Theta|\theta\rangle = \theta|\theta\rangle \qquad \langle\theta'|\theta\rangle = \delta(\theta'-\theta) \qquad -\pi \leqslant \theta', \theta \leqslant \pi \tag{22}$$

$$M|m\rangle = m|m\rangle \qquad \langle m'|m\rangle = \delta_{m',m} \qquad -\infty \leqslant m', m \leqslant \infty \tag{23}$$

$$\langle\theta|m\rangle = \frac{1}{\sqrt{2\pi m_0\theta_0}} \exp\left(\frac{\mathrm{i}\theta m}{m_0\theta_0}\right). \tag{24}$$

The cyclic notation becomes meaningless for the $|m\rangle$ states in the $N \to \infty$ limit, as this label becomes unbounded. For the $|\theta\rangle$ states, however, it takes naturally into account the boundary conditions that one good set of angle states must have, i.e.,

$$|\theta\rangle \equiv |\theta(\mathrm{mod}\, 2\pi)\rangle \tag{25}$$

[2] The author would like to thank one of the anonymous referees for drawing attention to this point.

and the action of the angle shift operator naturally obeys the boundary condition. But it has to be stressed that (as in the PB scheme) the range of the label θ is confined to $[0, 2\pi)$ by *definition*, and cyclicity modulo 2π is only a matter of notation. Therefore, and perhaps surprisingly, the usual results for angle/angular momentum variables are recovered from the discrete root from which the position–momentum results also emerged. Again, the product $m_0\theta_0$ must be set to $\hbar$. θ_0 is not expected to be a dimensional unit but must be related to how one is measuring the angle.

3. Conclusions

The basic aim here was to show that the two kinds of canonical variable defined on the basis of degrees of freedom *with* classical counterparts can be obtained from a description of a degree of freedom *without* a classical counterpart. In a pragmatic sense, one could say that the PB formalism for the angle/angular momentum case was seen as an extension of the Schwinger approach to quantum Cartesian variables. In addition, the discussion which led to those results has interesting aspects of its own.

One of those aspects is the role of the scaling factors in the limiting process. In the first part of the discussion, where the parameter δ is free to vary in the open interval $(0, 2)$, the initial discrete variables are changed to a position/momentum-like description, still discrete and with contour conditions holding prior to effectively considering the limit. The parameter δ controls the distance between successive eigenvalues of the Hermitian operators P and Q, and the greater one is, the smaller the other is—in such a way that their product is fixed. The infinite and continuum limit of these variables is the position–linear momentum pair. Schwinger had already stated that this would happen for $\delta = 1$, and what is surprising is that it happens for any value of δ in the open interval $(0, 2)$.

In the second part, we consider δ in one extreme of the interval previously considered ($\delta = 0$). Variables are now changed to an angle/angular-momentum-like description. The limit to the continuum in this case only affects one of the variables (in the discrete/continuous sense) and the angle/angular momentum operators and eigenstates are promptly recovered, basically reproducing the PB scheme. The first interesting thing is that, in this sense, angle/angular momentum variables are a limiting case of Cartesian variables and not the opposite. One also sees that, for a finite number of states, there is no fundamental distinction between angular or Cartesian coordinates, or—better—between the variables that will be identified with angular or Cartesian coordinates *after* the limit is taken, as representations (19) and (11) (prior to taking the $N \to \infty$ limit) can always be connected by a simple transformation. The possibility of this transformation is only lost after the limiting process has been carried out.

As a parallel remark, there is nothing in the simple steps that led from discrete to continuous variables that constrains the product of p_0q_0 to $\hbar$. In fact, there is no (technical) reason for this product to have the same value in the two situations. We know from experience that this happens, but it *could* be the case that $\hbar$ had a dependence on the number of states allowed to the system (but fortunately it seems that it does not).

In the sense above, one could say that it is not the geometry of a given system that imposes different quantum variables (in a quantization procedure over an infinite line or over a ring), but, rather, that there are different limiting cases of genuine discrete quantum descriptions that suit different geometries. The author cannot refrain from remarking that even the physical validity of this limit might be put into question [6]. After this work was finished, the author became aware of [7], which discusses in great detail a similar limit to the continuum, from a mathematical point of view.

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